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## Fuse model on a randomly diluted hierarchical lattice

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**Abstract.** The first breaking current of the randomly diluted fuse network in a hierarchical diamond lattice is studied theoretically and numerically. We show that the prediction proposed by Duxbury *et al* for Euclidean lattices should be applicable, and that it is correct: the average current necessary to break the first bond of the lattice decreases on average as the inverse of the logarithm of the system size. Its validity is shown to rely solely on the self-averaging of the conductivity with system size. Due to the peculiar geometry of the lattice, the breaking current as a function of the lattice size exhibits a series of plateaux followed by sudden variations between them. Once the first few bonds are broken, the rest of the breaking of the network is no longer comparable to that of a Euclidean lattice: the mean breaking current for the entire lattice decreases with system size much slower than the current necessary to break the first bond.

### 1. Introduction

A number of studies have been recently devoted to fracture models of disordered media, as reviewed in [1]. Various types of results have been obtained, and in spite of the simplicity of these models very different global behaviours have been reported, in particular concerning the extremely important size effect expected in these models. A classification of different behaviours has been recently proposed [2], providing a unifying picture of these models. However, a lot of work remains to be done to provide a firm theoretical basis to a collection of intensive numerical simulations.

The purpose of this paper is to investigate a simple type of disorder, dilution, that was first considered by de Arcangelis *et al* [3]. A fraction,  $q$ , of bonds in a lattice are missing at random. All the other bonds are present, and they have the same behaviour: When the current  $j$  they carry is smaller than 1, the bonds acts as ohmic conductors, with a conductance 1. When  $j$  reaches 1, the bond breaks, and it becomes irreversibly an insulator.

Such a disorder has been previously studied, by Duxbury *et al* [4]. We will recall below their theoretical prediction which leads to a decrease of the failure current density with system size  $L$  as a power law of the logarithm of  $L$ . Then this argument has

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been used again in a variety of models which differ by the local behaviour: fuses with a residual conductivity [5], dielectric breakdown [4], brittle springs [6], superconductors [7], Born model [8], etc.

However, the original argument by Duxbury *et al* (hereafter referred to as the DBL argument) contains a few approximations that are difficult to test directly. Indeed, a logarithmic decay of the mean threshold current is extremely difficult to test accurately. In this paper, we consider a simplistic geometry, namely that of a hierarchical diamond lattice. This geometry, together with some simple share loading rules, has been already considered by Sornette [10]. For this lattice, we will show that in the case of a random dilution, all the conditions necessary to apply the same argument as for Euclidean lattices are fulfilled. The simplicity of the geometry allows us to investigate a considerable range of system sizes:  $2^{17}$ , or more than 5 decades between the smallest and the largest system, and this without any approximation. Furthermore, we will present a simplification, the constant conductivity approximation (CCA), which allows us to account for all observed features. This will allow us to test the theoretical prediction and show that it gives a reliable description of the breaking of the first bond, but not of the entire system.

## 2. The DBL argument

The argument developed by Duxbury *et al* [4] consists of two parts: the first one is to identify the current,  $I_c$ , that is necessary to break the very first bond in the system. The second part is the hypothesis that once one bond is broken, then the current to break a second, a third, ...,  $n$ th bond will be smaller than  $I_c$ , and thus  $I_c$  will be the breaking current for the entire network. We will discuss the second hypothesis later on. However, the hierarchical lattice is likely to exhibit a behaviour different from that of a Euclidean lattice once many bonds are broken. Most of the rest of this paper is devoted to the derivation of  $I_c$  for the breaking of the first bond.

The analysis of Duxbury *et al* for computing  $I_c$  is twofold. The first step (A) is to identify the weakest part in the lattice, in the form of a large size defect of size  $l$ , and to compute the maximum current flowing at its tip as a function of its size. The second step (B) is to compute the size of the largest defect  $l^*(L)$  one can encounter in a lattice of size  $L$ .

A. Following [3], we choose as a typical defect a linear crack of size  $l$ , i.e.  $l$  consecutive aligned bonds. For such a defect placed in a uniform homogeneous medium, the maximum current flowing in the network is located at the tip of the crack. This maximum current  $j(l)$  is proportional to the size of the defect to the power  $\frac{1}{2}$  (see [9] for a discussion of the finite size effects concerning this relation):  $j(l) \propto \sqrt{l}$  for an input current  $I_0 = L$  such that  $j$  would be equal to 1 for a defect-free system. Choosing another defect geometry, or moving to a three-dimensional medium, or changing the local behaviour of the bonds may lead to a different power law; however, the key point is the existence of a positive exponent  $\alpha$  such that

$$j(l) \propto l^\alpha. \quad (1)$$

B. We now need to compute the probability  $p(l)$  that such a defect of size  $l$  is present in a lattice of size  $L$ . Neglecting the environment of this defect leads to  $p(l) = q^l$  where

$q$  is the probability that one bond is missing. Now to estimate the typical size  $l^*(L)$  of the largest defect one encounters in a lattice of size  $L$ , we use the following result:  $p(l^*(L)) = L^{-2}$  in a two-dimensional lattice. This simply means that the probability of the defect  $p(l^*(L))$  times the number of places where this defect can occur should be of order 1. Thus

$$l^*(L) = 2 \frac{\log(L)}{\log(1/q)}. \quad (2)$$

Combining the two results A and B, one obtains a maximum current flowing in a lattice of size  $L$

$$j_{\max} \propto \left( \frac{\log(L)}{\log(1/q)} \right)^\alpha. \quad (3)$$

The external current for which this bond will reach its threshold of 1, is the inverse of  $j_{\max}$  times  $I_0$ . Thus the conclusion of Duxbury *et al* is

$$I_c(L)/L \propto \left( \frac{\log(L)}{\log(1/q)} \right)^{-\alpha}. \quad (4)$$

Some numerical computation reported in [3] shows that the current for which the first bond burns does indeed decrease with  $L$  logarithmically, with an exponent  $\alpha \approx 1$ , although the determination of this exponent is quite difficult.

Let us comment a little more on some implicit hypotheses that were used in the derivation of the DBL argument as recalled above. In order to compute the maximum current flowing around a defect, one has to assume that the environment of the defect is a homogeneous lattice. To take into account defect interactions in the general case is too complex for one to reach any reliable conclusion. Such a task can only be performed in the case of simple configurations of two defects, and it does not change the basic result (1) more than by different values of  $\alpha$ .

In contrast to this, in step B, for the computation of the probability of the occurrence of a defect, one makes no assumption as to the environment of the crack. Indeed, the hypothesis that the rest of the medium is defect free would introduce a multiplicative contribution to the probability of the order of  $p^{L^d}$ , which combined with the defect probability would prevent one from deriving the result (2).

The two contradictory assumptions implicit in the two steps of the derivation imply that the result (4) is not proved. It also seems difficult to improve the argument.

### 3. Application to the hierarchical lattice

Figure 1 presents the construction of the hierarchical lattice. We proceed recursively starting from one single bond. Then at the next generation, we replace all bonds of the previous generation by a diamond of four bonds. Figure 1(a) shows the basic recursion, and figure 1(b) shows the lattice obtained at the fourth generation. Although this lattice has not the topology of a Euclidean lattice, its dimension is equal to 2. Thus it provides a simple example of a two-dimensional medium, on which many properties expected for Euclidean two-dimensional lattices can be tested, since the simplicity of

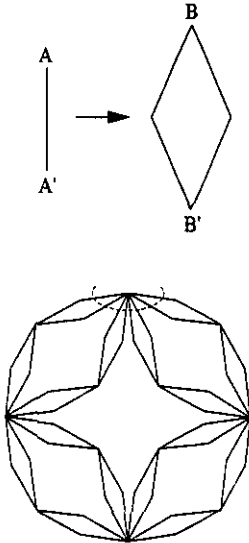


Figure 1. Top: construction of the hierarchical lattice. The construction of this lattice consists of changing every single bond into a diamond of four bonds when going from one generation to the next. Bottom: hierarchical diamond lattice shown at the fourth generation.

the geometry allows one to consider much larger sizes numerically, and to derive some analytical results.

What are the ingredients necessary to apply the argument developed above? In order to get the result A, one needs an increase of the maximum current as a power law of the defect size, in an infinite size system.

In appendix A, we give all the details of the computations of the maximum current flowing in a system at any generation, with one single crack of length  $l$ . We report in figure 2(a) this evolution in a lattice of size  $2^{10} = 1024$ . We see a complex structure which certainly cannot be described by a single power law. The fact that this set of values does not show a monotonic increase in contrast to what happens in a Euclidean lattice has no significant effect on the validity of the comparison between these two types of lattices concerning the characteristics of the breaking of the first bond, but it will affect the comparison for the rest of the evolution in the breaking process. As will be discussed later, we do not expect the whole breaking process to be strictly comparable in these two lattices.

All data points are, however, confined to two power laws with exponents 0 and 1. We also note the following result: the fraction of cracks whose current increases faster than a given power law of  $l$ ,  $l^\alpha$ , (or equivalently the fraction of points in figure 2(a) that are above a line of slope  $\alpha$ ) is a continuous function of  $\alpha$  that goes from 1 to 0 as  $\alpha$  increases from 0 to 1. Let us consider the CCA developed in appendix A. For a crack of length  $l$  the maximum current flowing in the network can be well approximated by a constant times  $2^{x(l)}$  where  $x(l)$  is the number 1 in the binary representation of  $l$ , as discussed in appendix A. Such an approximation leads to the evolution of the maximum current as shown on figure 2(b), in good agreement with the previous result, when the defect size is small compared with that of the system,  $l \ll L$ . For  $2^{n-1} \leq l < 2^n$ , the number of crack lengths that satisfy  $j(l) = 2^k$  is equal to  $\binom{n-1}{k-1}$ .

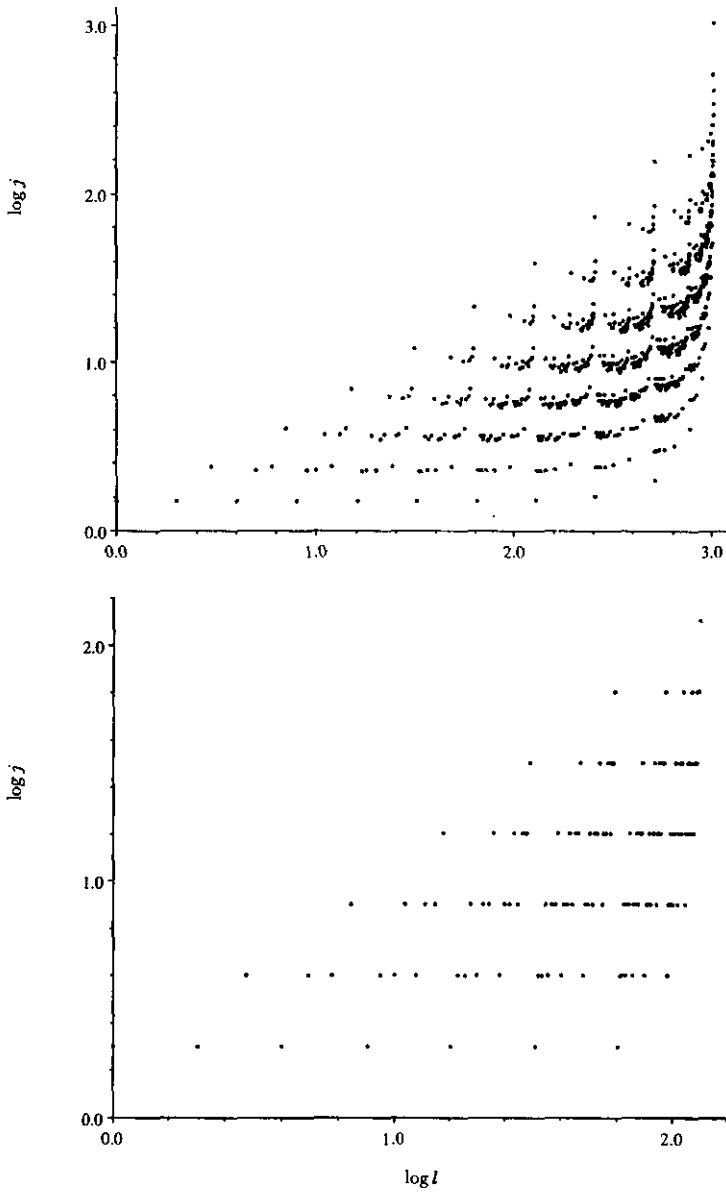


Figure 2. (a) Evolution of the maximum current  $j$  in a lattice at the tenth generation as a function of the crack length  $l$ , on a log-log scale. The location of the crack is along the dotted line in figure 1, starting from one border. (b) As (a), but using the constant conductivity approximation. A good agreement is found between both cases when the crack length is small compared with the system size.

The mean value of  $j(l)$ ,  $\langle j(l) \rangle$ , is thus

$$\langle j(l) \rangle = \langle 2^{z(l)} \rangle = 2^{1-n} \left( \sum_{k=1}^n \binom{n-1}{k-1} 2^k \right) = 2 \left( \frac{3}{2} \right)^{n-1} \quad (5)$$

Thus the maximum current increases on average as  $l^{\alpha_1}$  where  $\alpha_1 = \log(\frac{3}{2})/\log(2) \approx$

0.585. However, very few crack lengths contribute to this average. Indeed, on average

$$\langle x(l) \rangle = 2^{1-n} \left( \sum_{k=1}^n \binom{n-1}{k-1} k \right) = \frac{n+1}{2} \tag{6}$$

thus  $2^{\langle x(l) \rangle} = 2^{(n+1)/2}$  indicating an increase of the current as  $l^{\alpha_2}$  with  $\alpha_2 = \frac{1}{2}$ . One can go further in this analysis and show that the distribution of the maximum current for a single crack is multifractal: The number of crack lengths for which  $j$  increases as  $l^\alpha$ , scales as  $l^{f(\alpha)}$ , where  $0 \leq \alpha \leq 1$ .  $f$  reaches its maximum,  $f = 1$ , for  $\alpha = \frac{1}{2}$ , and it drops to zero symmetrically either side of  $\alpha = \frac{1}{2}$ , when  $\alpha = 0$  or  $1$ . More precisely, after some simple calculations (expanding the binomial coefficient using Stirling's formula)  $f(\alpha)$  can be written as

$$f(\alpha) = \log_2(\alpha^{-\alpha}(1-\alpha)^{-(1-\alpha)}). \tag{7}$$

Let us return to the application of step A in the DBL argument. Let us now choose arbitrarily a value  $\alpha_0$  strictly less than  $\frac{1}{2}$ , and strictly positive. We disregard all crack configurations for which  $j$  increases less quickly than  $l^{\alpha_0}$ . The fraction of the configurations that we reject scales as  $l^{f(\alpha_0)-1}$ . Since  $\alpha_0 < \frac{1}{2}$ , this proportion tends to zero with  $l$ , and thus most configurations are kept. With this argument we have justified the form A in the case of the hierarchical diamond lattice with a strictly positive exponent  $\alpha$ . It suffices now to reproduce step B of the argument with the selected subset, and thus to obtain the result of Duxbury et al [4].

#### 4. Solution using the CCA

For all dilutions such that the network is above its percolation threshold ( $p > p_c = (\sqrt{5} - 1)/2$ ), the conductance of the network converges rapidly toward an asymptotic value, as the system size increases. More precisely, the width of the distribution of the conductances of lattices of size  $L$  decreases as  $L^{-1}$ . This convergence allows one to simplify the complete treatment of the evolution of the maximum current which in principle should be treated jointly with that of the conductance. This simplification, presented in appendix A for the case of a single crack and in appendix B for a random dilution of the network, is referred to as the constant conductivity approximation, or CCA. The fluctuations of the conductance around the mean value are neglected, and the conductance of conducting networks is taken to be the same for all networks. As the fraction of broken bonds tends to zero, or as the lattice size tends to infinity, this approximation comes closer to the real solution.

In appendix B, we present the derivation of the probabilities  $\pi(k, n)$  that the maximum current in a lattice at generation  $n$  is smaller than or equal to  $2^k$ . Indeed, within the CCA, the maximum current can only assume values that are integer powers of 2. We report here the basic recursion formula:

$$\pi(k, n) = P_2(n)\pi(k-1, n-1)^2 + (1-P_2(n))\pi(k, n-1)^4. \tag{8}$$

Together with the initial condition,  $\pi(k, 0) = 1$  for all  $k$ , and with the expression of the probability  $P_2(n) \approx (4q_0)^{2^n}$  (see equations (B5) and (B6)), this last equation determines the evolution of the mean maximum current

$$\langle j(n) \rangle = 2^n - \sum_{k=0}^{k=n-1} \pi(k, n) 2^k. \tag{9}$$

Let us note the following properties of  $\pi(k, n)$ . For  $k > n$ ,  $\pi(k, n) = 1$  since it is not possible to generate a current larger than  $2^n$  in a lattice of generation  $n$ . At fixed  $n$ ,  $\pi(k, n)$  increases with  $k$ , and for fixed  $k$ ,  $\pi(k, n)$  decreases with  $n$  since the maximum current cannot decrease with system size.

We will study the evolution of  $\pi(k, n)$  as a function of  $n$  at fixed  $k$ . The two contributions of (8) play very different roles. Indeed the first one is only important in initiating a value of  $\pi(k, n)$  different from 1, and setting a gap between different  $k$ , as we will see below.

When  $\pi(k, n)$  is close to 1, as expected for small  $n$ , we write  $\pi(k, n) = 1 - \epsilon(k, n)$ . Equation (8) gives

$$\epsilon(k, n) = 2 P_2(n) \epsilon(k - 1, n - 1) + 4 \epsilon(k, n - 1) \tag{10}$$

where we have neglected  $P_2(n)$  assuming  $P_2(n) \ll 1$  as can easily be shown. Since  $\epsilon(k, n)$  increases with  $n$ , and  $P_2(n)$  decreases exponentially with  $n$ , we are led to neglect the first contribution in the right-hand side of (10) as soon as  $\epsilon(k, n - 1)$  is non-zero. This leads to

$$\epsilon(k, n) \approx 4^{n-k-1} \epsilon(k, k + 1). \tag{11}$$

The first non-zero value  $\epsilon(k, k + 1)$  is determined by the first term of the right-hand side of (10); which when iterated yields

$$\begin{aligned} \epsilon(k, k + 1) &= 2 P_2(k + 1) \epsilon(k - 1, k) = 2^k \left( \prod_{i=2}^{i=k+1} (4q_0)^{2^i} \right) \epsilon(0, 1) \\ &= 2^k (4q_0)^{2^{(k+2)}-1} (1 - (4q_0)^2) \end{aligned} \tag{12}$$

where  $P_2(k) = (4q_0)^{2^k}$  from (B5) and (B6). Equation (10) is only valid when the condition  $\epsilon \ll 1$  is fulfilled. To obtain this limit quantitatively, we can compute the generation  $n^*(k)$  for which  $\epsilon(k, n)$  is of order 1, namely

$$2(n^*(k) - k - 1) + k + (2^{(k+2)} - 1) \log_2(4q_0) = \mathcal{O}(1) \tag{13}$$

as obtained from (12) substituted in (11). This gives the expression of

$$n^*(k) \approx 2^k \log_2(1/4q_0) \tag{14}$$

for large  $k$ . Above this generation,  $\pi(k, n)$  is expected to be small compared with 1. We can again use equation (8), and neglect the first term since  $\pi(k, n^*(k))$  is of order 1, and  $P_2(n)$  decreases exponentially. We obtain

$$\pi(k, n) \approx \pi(k, n - 1)^4 \tag{15}$$

or

$$\pi(k, n) \approx \pi(k, n^*(k))^{4^{n-n^*(k)}}. \tag{16}$$

The probability that the maximum current is exactly equal to  $2^k$  is equal to  $\pi(k, n) - \pi(k - 1, n) = \varpi(k, n)$ . Thus, when  $\pi(k, n)$  is close to 1,  $\varpi(k, n) = \epsilon(k - 1, n)$ , and when  $\pi(k, n)$  is close to zero,  $\varpi(k, n) \approx \pi(k, n)$ .



To summarize, starting from an extremely small value (a small number raised to the power  $2^k$ , cf (12)),  $\varpi(k, n)$  increases with  $n$  exponentially, for a number of generations ( $\approx n^*$ ), exponentially with  $k$  (equation (14)) and then  $\varpi(k, n)$  decreases as an exponential of an exponential (equation (16)).

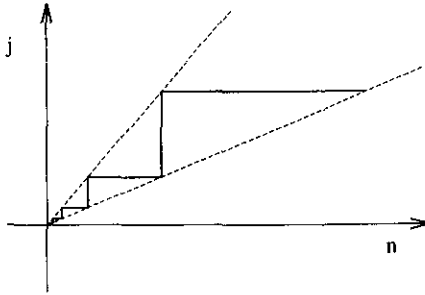
From this description of the evolution of  $\pi(k, n)$ , we can deduce the scaling of  $\langle j(n) \rangle$ . The distance between two consecutive probabilities  $\varpi(k, n)$  when  $n$  increases by one becomes so large as  $n$  increases that only one dominates the others. The dominant  $\varpi(k, n)$  at fixed  $n$  is the one for which  $n$  is close to  $n^*(k)$ . The mean maximum current is thus  $\langle j \rangle \approx 2^k$ , with

$$k = \log_2 \left( \frac{n}{\log_2(1/4q_0)} \right) \quad (17)$$

using (14) to obtain  $k$  as a function of  $n$ . Setting this value back into the expression of the current, we obtain

$$\langle j \rangle \approx \frac{n}{\log_2(1/4q_0)} \quad (18)$$

and thus a mean maximum current that is proportional to  $n = \log_2(L)$ , i.e. the result of Duxbury *et al* [4]. Note, however, that the result (18) has not been obtained by a reasoning parallel to that of [4]. We have instead worked directly on the distribution of currents.



**Figure 3.** Schematic plot of the maximum current as a function of the generation using the CCA. Since the maximum current can only be an integer power of 2, we expect to see a discrete set of steps confined between two linear behaviours. The mean increase of the current is thus linear with the generation as expected from the DBL argument.

To complete our analysis, we should, however, notice an artefact of the hierarchical lattice, i.e. the fact that the maximum current can only assume integer powers of 2. Thus the  $k$  determined through (17) can only be an integer. This will produce a rather surprising effect sketched in figure 3: namely that the mean maximum current as a function of the generation will display steps of larger and larger width and height. We also note that this is specific to the CCA. Considering the problem without any approximation will generate in the first iteration a distribution of maximum current with a certain width. This statistical spreading of the data is expected to restore a smooth mean trend similar to that of (18). However, as the lattice size increases, we become closer and closer to the hypothesis of the CCA, and thus the discreteness of

$k$  will appear. This size effect is expected to be smaller in the case of a small initial value of  $q_0$ .

Let us note here an additional similarity between the hierarchical lattice and a Euclidean lattice. Li and Duxbury [11] have shown that the current distribution presents an exponential tail for large currents. This observation allows us to rederive the variation of the maximum current as a function of the system size. For a hierarchical lattice, the current distribution (and not of the maximum current) presents also an exponential decay for large currents. The way to have access to this information is to correct the recursion relation (8) in order to erase the selection of the maximum element in the writing of this relation. The fact that the maximum current was selected previously introduced the exponentiation of  $\pi$  in (8) to the second and to the fourth power. The probability  $\mu(k, n)$  that a current is larger than or equal to  $2^k$  in a lattice at the  $n$ th generation satisfies a *linear* relation:

$$\mu(k, n) = P_2(n)\mu(k-1, n-1) + (1 - P_2(n))\mu(k, n-1). \quad (19)$$

It is a simple matter to show that for fixed  $k$ , the expression of  $\mu(k, n)$  can be written for large  $n$  as

$$\mu(k, n) \propto \left( \prod_{i=0}^{i=k} P_2(i) \right) \approx (q_0)^{2^k} \quad (20)$$

where the current  $j$  is equal to  $2^k$  as before. This probability is simply that of encountering a defect of length  $l = 2^k$ . Thus the current distribution is proportional to  $q_0^j$ , as expected for an Euclidean lattice.

Duxbury and Leath [12] used the previously recalled result in order to obtain the form of the survival probability for a lattice of size  $L$  subjected to a current  $I$  using extreme statistics theory. The probability,  $P(I, L)$ , that no bond breaks under the application of a current density  $I$  assumes the following form:

$$P(I, L) \propto \exp(-aL^d \exp(-bL/I)) \quad (21)$$

where  $a$  and  $b$  are constants. Obviously for a hierarchical lattice such a form is also expected, since it is equivalent to the law of decrease of the breaking current of one bond. This result relies on the hypothesis that the current variable can be assumed not too much correlated.

At the end of section 2 we have criticized the derivation of the DBL argument noting that it was local and that it neglected the environment of the crack. We have seen, however, that in the case of the hierarchical lattice, both steps A and B can be satisfied due to the fact that the CCA holds. In this case, since we can neglect the influence of the main crack on the conductivity of the lattice, the maximum current occurs as a result of a local environment, and it is not necessary to describe accurately the geometry of sublattices whose size is larger than the crack size. This makes the DBL analysis valid in this case. The transposition to Euclidean lattices is certainly correct since the basic sufficient ingredient has been shown in our study to be the self-averaging of the conductivity with system size, a generic property of any lattice of dimension larger than 1, as soon as the lattice is not close to the percolation threshold.

## 5. Numerical computations

In order to test the above analysis, we have performed some numerical simulations of the maximum current as a function of the generation of the lattice. These simulations did *not* involve any approximation such as discussed previously. We only used the hierarchical structure of the lattice. We record for each lattice both the conductance and the maximum current flowing through it for a total input current equal to unity. As soon as four lattice of the same generation were generated we computed the pair  $(g, j)$  of the lattice at the next generation, and we erased the four previous pairs. This procedure allows us to store only a small amount of information, although the maximum size of the systems generated were rather large,  $2^{17}$ . At most four pairs of numbers have to be stored at each generation. The limiting factor of these simulations is the computation time.

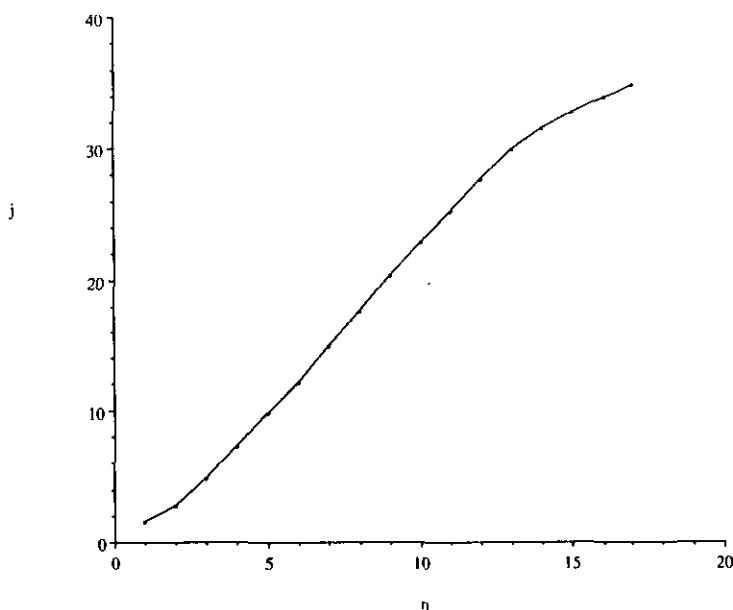


Figure 4. Average maximum current flowing in a lattice for a dilution  $p = 0.8$ , as a function of the generation of the lattice,  $n$ , or equivalently the logarithm of the system size  $n = \log_2(L)$ .

We studied two dilutions:  $p = 0.8$  and  $p = 0.99$ . The first value was chosen so as to be simultaneously far from the percolation threshold, which is in this case  $p_c = (\sqrt{5} - 1)/2 \approx 0.618$ , and from  $p = 1$ . The second value of  $p$  was chosen so as to see sooner the large size behaviour (see (14)). In both cases, we generated one lattice of the 18th generation, and thus four lattices at generation 17,  $\dots$ , and  $4^{17}$  at the first generation.

Figures 4 and 5 shows the average maximum current as a function of the generation (or equivalently the logarithm to base 2 of the system size), for both dilutions. Two regimes are clearly visible on these plots: For small system sizes, (generation less than 14 and 12 respectively)  $j$  increases linearly with the logarithm of the size, in agreement with the DBL result with an exponent  $\alpha$  appearing in (1) equal to 1. In

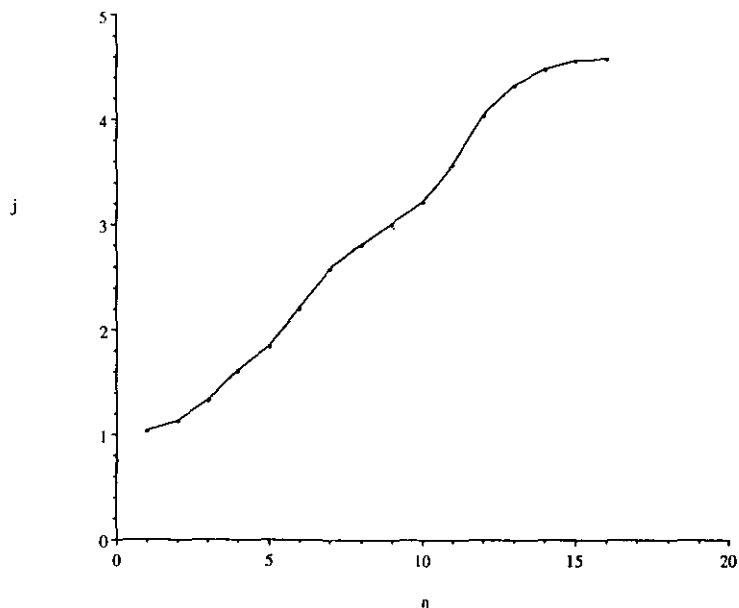


Figure 5. As figure 4, but for a dilution  $p = 0.99$ .

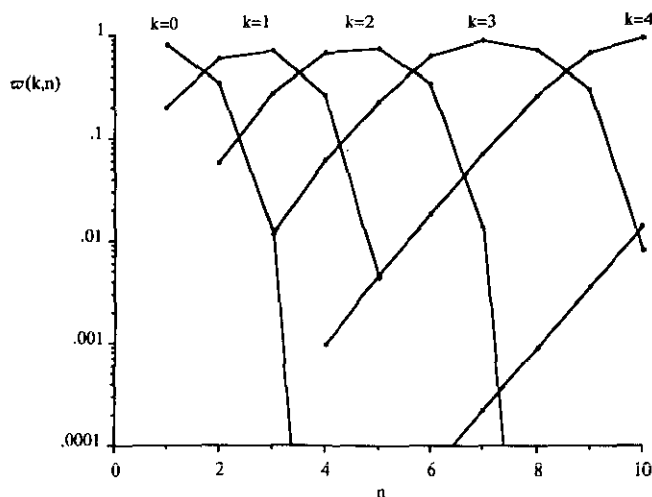
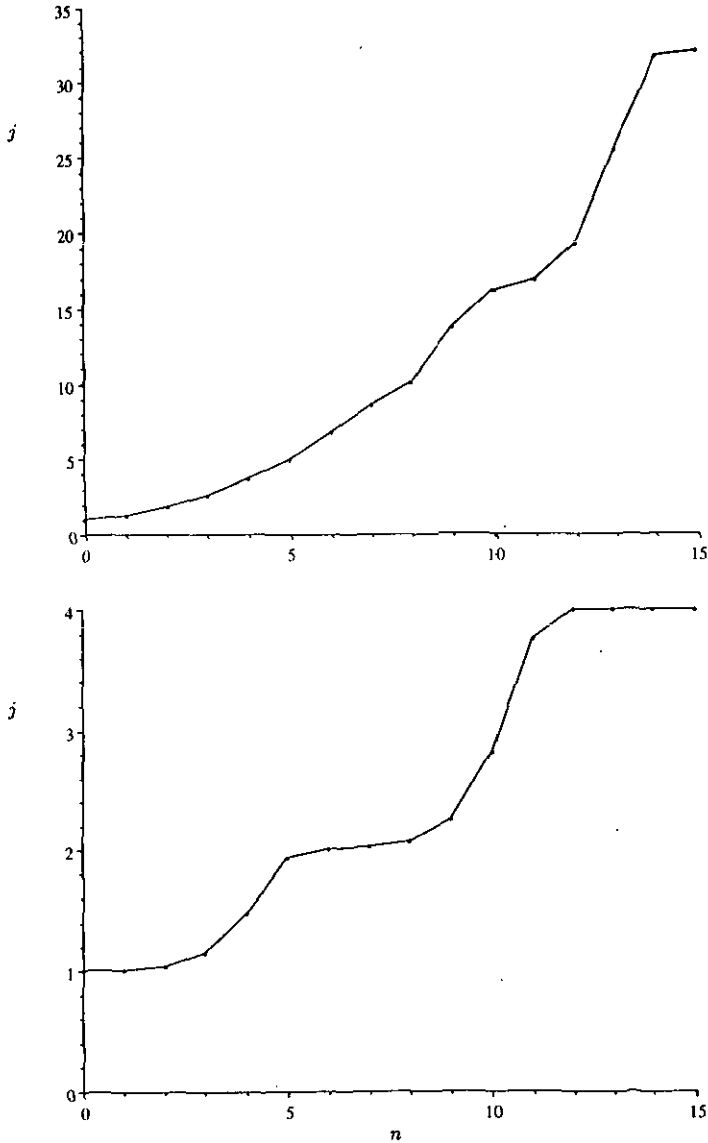


Figure 6. Computed probability  $w(k, n)$  that the maximum crack is of size  $2^k$  in a lattice at generation  $n$ , as a function of  $n$  using the CCA. The first five values of  $k$  are shown on this plot.

contrast, for large system size  $j$  departs markedly from the previous linear behaviour.  $j$  seems to saturate at a constant value. Let us note that already for small sizes in the case of  $q_0 = 0.01$ , figure 5, we can see some oscillations in the data points. These oscillations are not due to poor statistics, but on the contrary they are a systematic effect: the steps that were obtained in the evolution of  $j$  with the CCA, are indeed expected to be more and more visible as  $L$  increase and for small  $q$ . Similarly, the



**Figure 7.** (a) As figure 4, but using the CCA. We see the discrete steps more clearly than in figure 4, but the mean trend is similar. (b) As figure 5, but using the CCA. For values of the initial dilution  $q_0$  as small as 1%, the agreement with the real values is better, and begins for smaller sizes.

apparent saturation of the maximum current is also the manifestation of the discrete nature of the possible maximum current. For  $n > 17$ , figure 5 will show larger and larger oscillations, as sketched in figure 3, with however an average linear increase as predicted from (18).

In order to see this point more clearly, we have performed an analogous simulation, considering the CCA. The difference with the argument developed in the previous paragraph, is that we did not simplify the expression of the various probabilities (in particular  $P_2(n)$ ).

Figure 6 shows the evolution of the probabilities  $\varpi(k, n)$  as a function of  $n$  for

the first few values of  $k$  for the case  $q_0 = 0.2$ . On a semi-log scale, we see that these probabilities tend to display the properties that were obtained in the previous paragraph, when  $k$  is larger than 2 or 3. In a first stage, the probability  $\varpi$  increases as an exponential of  $n$ , then reaches a maximum, and finally decreases abruptly as an exponential of an exponential. We also see that these functions have approximately the same shape for different values of  $k$  and one can be deduced from another by a simple translation along the  $n$  axis. The mean distance between these functions counted along the  $n$  axis increases with  $k$ , as can be expected from (14), which gave the position of the maximum as a function of  $k$ .

Figures 7(a) and 7(b) shows the evolution of the mean maximum current as a function of the generation using the CCA for  $q_0 = 0.2$  and 0.01, respectively. We see distinctly the steps that were sketched in figure 3. Comparison between figures 4 and 7(a), and between figures 5 and 7(b), indeed shows that the CCA becomes more and more valid as the lattice generation increases, as previously expected. This allows us to conclude that for large system size, the evolution of the maximum current is expected to follow that sketched in figure 5, although a direct simulation would rather suggest a saturation effect.

## 6. Complete fracture of the lattice

Finally let us note that although we believe the results exposed above concerning the features of the first broken bond to be fairly general, it is not the case for the whole evolution of the fracture. The reason for such a difference is the presence of holes of all sizes in the hierarchical lattice. Thus, if we imagine a single crack developing in a homogeneous medium, the series of breaking currents at each step of the propagation of the crack will reproduce the series shown in figure 2(a).

Thus, we see that a simple linear growth of a crack is not a likely scenario: the maximum current would in this case decrease to values as low as 2, whenever the crack encounters a hole of its size, i.e. a trap. Due to the hierarchical construction of the lattice, holes of all sizes are present in the lattice, as can be seen from figure 1. This can be likened to the classical way of stopping crack propagation by cutting a circular hole at the crack tip, so as to eliminate the singularity of the stress at this point.

What will happen can, however, be guessed rather straightforwardly. First the dominant crack will grow until it reaches a hole that makes its maximum current as low as that of the second defect present in the initial geometry. This process will then take place at all scales until all cracks have made their way to a trap. Thus we expect that the number of bonds to be broken at the end of the breaking process will be of the order of  $L^2$  (up to logarithmic corrections). Some numerical simulations of the complete fracture process, and not only for the first bond, can easily be performed. However, these are much more demanding in terms of computer time. Going up to the tenth generation indicates that the number of broken bonds when the current reaches its maximum,  $N_c(n)$ , increases with the generation  $n$  of the lattice more quickly than  $L^{1.70}$  (where  $L = 2^n$ ): the log-log plot of  $N_c(n)$  against  $L$  is strongly curved upward, and thus the tangent slope for  $n = 10$  is certainly a lower bound for the true exponent, in agreement with the expectation 2.

In fact, under the assumption of the validity of the CCA during the entire fracture process, the problem has been solved analytically by Gabrielov and Newman [13]. These authors indeed considered such a hierarchical lattice with a distribution of

strength for each bonds, together with a load sharing rule for the network that is in fact identical to that used in the CCA. Gabrielov and Newman [13] did not try to justify this approximation, but they obtained the very general conclusion that the mean breaking current for the entire lattice decreased with system size as

$$I_c(L)/L \propto 1/\log(\log(L)) \quad (22)$$

or a decrease much slower than that related to the breaking of one single bond as given in (4). The method they used is rather different from our approach. It is based on a renormalization equation of the full probability distribution for a complete failure of the system. It thus takes the effect of traps mentioned above fully into account. Our approach has shown the validity of the CCA for random dilution. However, one would need a stronger property, namely the validity of the CCA during the entire fracture process in order to apply directly the proof of Gabrielov and Newmann to our system.

Let us mention that for Euclidean lattices, the trapping of cracks into holes is not expected. However, some preliminary simulations on two-dimensional square lattices shows that the entire network does not break down as soon as one bond is broken. The increase of  $N_c(L)$  is however not as strong as in the case of the hierarchical lattice ( $L^2$ ), but rather  $N_c(L) \propto L \log(L)$  [14]. This last observation seems to indicate that the argument derived for the breaking of the first bond cannot be used to determine the breaking of the entire lattice. Thus the problem of the entire fracture is still open, until any firm conclusion concerning the number of broken bonds at maximum force is found.

## 7. Conclusion

We have seen that the evolution of the maximum current encountered in a randomly diluted hierarchical lattice exhibits the mean scaling features expected for Euclidean lattices, as predicted by Duxbury *et al* [4]. However, The geometry of the lattice produces an increase of the maximum current with system size by discrete steps.

The breaking of the whole lattice is expected to give different results for a hierarchical and a Euclidean lattice.

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## Appendix A. Computation of the maximum current for a single crack

All information needed to characterize the maximum current in a lattice of any size and structure is given by the pair  $(g, j)$ , where  $g$  is the conductance of the lattice and  $j$  the maximum current density that flows in the medium for an input current that would give a unit current in all bonds if the lattice were intact.

Due to the symmetries of the diamond lattice, we may locate the crack in the top row of figure 1, as shown by a dotted line. If we consider a crack of length  $l$  in a medium of size  $L = 2^n$ , two cases can occur: either the crack is shorter than  $L/2$  (case 1) or it is longer (case 2).

For case 1, let the pair  $(g, j)$  be relative to the sublattice (at generation  $(n - 1)$ ) that contains the whole crack. It is straightforward to compute the pair  $(g', j')$  for the entire lattice at the  $n$ th generation. After some simple calculations, the transformation can be written as

$$\begin{aligned} g' &= \phi_1(g) = \frac{1 + 3g}{2(1 + g)} \\ j' &= \psi_1(g, j) = \frac{4gj}{(1 + 3g)}. \end{aligned} \tag{A1}$$

For case 2, one sublattice is cut by the crack, whereas the sublattice in parallel contains the tip of the crack. We can again compute  $(g', j')$  to be

$$\begin{aligned} g' &= \phi_2(g) = \frac{g}{(1 + g)} \\ j' &= \psi_2(j) = 2j. \end{aligned} \tag{A2}$$

Let us first study the conductance of the lattices, using the fact that  $\phi_1$  and  $\psi_1$  only depend on  $g$ . For a crack of length 1, we are in case 1 at all generations of the lattice. Thus the conductance at the  $n$ th generation,  $g_n$ , is obtained by  $n$  successive iteration of the function  $\phi_1$ ,  $g_n = \phi_1^{(n)}(0)$ , where the superscript  $(n)$  stands for iterations. It is trivial to show that  $g_n$  converges towards the only fixed point of  $\phi_1(x)$ , i.e. 1, exponentially with  $n$ . For large enough  $n$ ,  $g_n$  can be written

$$g_n \approx 1 - A4^{-n}. \tag{A3}$$

Due to the symmetry of the lattice, we can always choose to start the crack from the left side of the top row. For a crack length  $l$  larger than 1, we use the binary representation of  $l$  as  $l = a_i a_{i-1} \dots a_1 a_0$ . For any order  $j$ , if  $a_j = 1$ , then the sublattice at the  $j$ th generation is cut (case 2) and thus we need to use the function  $\phi_2$ . If  $a_j = 0$  then the sublattice in parallel with the crack is intact and thus we have to use  $\phi_1$ . Combining the successive cases relevant to the crack geometry, the conductance at the  $n$ th generation, for  $n \geq i + 1$  is given by

$$g_n = \phi_{k_n}(\phi_{k_{n-1}}(\phi_{k_{n-2}}(\dots \phi_{k_0}(0)\dots))) \tag{A4}$$

where  $k_j = a_j + 1$ . Since for all index larger than  $(i + 1)$  the crack is confined into a sublattice that will always be in case 1,  $a$  is zero, or  $k$  is 1, we will finally always be in a situation similar to that of a crack of length 1, and thus  $g_n$  always converges to 1 exponentially with  $n$  as in (A3).

In order to obtain the maximum current as well as the conductance, the computation is quite similar. The series of functions to iterate is given again by the binary writing of  $l$ . Such a computation gives the evolution of the maximum current as a function of its length in a lattice of size  $2^{10}$ , as shown in figure 2(a). However, the fact



that  $j'$  depends both on  $j$  and  $g$  makes the computation a little more complicated. In order to obtain a simpler expression for the maximum current, we now introduce an approximation: the constant conductivity approximation (referred to throughout this paper as CCA).

Indeed the conductance approaches unity extremely fast with  $n$  (see (A3)), as soon as the system size is larger than the crack length. Thus we propose to set  $g_n$  equal to its asymptotic value of 1, in the computation of the maximum current  $j_n$  at the  $n$ th generation. With this assumption, the function  $\psi_1(j, g)$  simplifies to  $\psi'_1(j) = \psi_1(j, 1) = j$ . For case 2, the function  $\psi_2(j) = 2j$  obviously remains unchanged.

The very simple form of the resulting transformations  $\psi'_1$  and  $\psi_2$  allows one to write down explicitly the expression of the maximum current using the CCA. For a crack of length  $l$  we use an equation similar to (A4) using  $\psi$  instead of  $\phi$  functions. We obtain simply

$$j_n(l) = j_0 2^{a_n} 2^{a_{n-1}} \dots 2^{a_0} = j_0 2^{x(l)} \tag{A5}$$

where the function  $x(l)$  is the number of 1s in the binary writing of  $l$ . We should note that (A5) is now independent of  $n$ , since increasing  $n$  simply consists of iterating the function  $\psi'_1(j) = j$ .

In simple terms, within this approximation, every times a sublattice shares its current with another one, the maximum current stays constant, whereas when it is alone, its current is doubled. Figure 2(b) shows the simple result (A5) compared with the real solution already displayed in figure 2(a). We see a good agreement for system sizes much larger than the crack length. Introducing a free scale factor in the CCA result for  $j_n$ , (i.e. choosing  $j_0$  at the zeroth generation in (A5) as a free parameter) allows us to obtain both an upper and a lower bound for the true solution. Thus the scaling properties we can extract from this approximate solution are also valid for the solution without approximations.

Let us note finally that the real interest of the CCA approximation lies in the more complex case of the random dilution studied in greater detail in appendix B.

### Appendix B. The random dilution case

In this appendix, we derive the probability of encountering a given maximum current as a function of system size in a randomly diluted hierarchical lattice.

As in the appendix A, we note that each lattice, at any generation, can be characterized simply by the pair  $(g, j)$  where  $g$  is the conductance of the network, and  $j$  is the maximum current flowing through it for a unit external current. Going from one generation to the next consists of computing the resulting pair  $(g', j')$  after having combined four sublattices  $(g_1, j_1)$ ,  $(g_2, j_2)$ ,  $(g_3, j_3)$ , and  $(g_4, j_4)$ , first in series two-by-two, and then in parallel. Thus when all four lattices are conducting (case 1)

$$g' = \frac{1}{1/g_1 + 1/g_2} + \frac{1}{1/g_3 + 1/g_4} = \Phi_1(g_1, g_2, g_3, g_4) \tag{B1}$$

$$j' = 2 \max \left( \frac{\max(j_1, j_2)}{g'(1/g_1 + 1/g_2)}, \frac{\max(j_3, j_4)}{g'(1/g_3 + 1/g_4)} \right) = \Psi_1(g_1, g_2, g_3, g_4, j_1, j_2, j_3, j_4)$$

using the same notation as in appendix A. If one branch does not conduct (say  $g_3 = 0$  or/and  $g_4 = 0$ ), whereas the other one is not cut (case 2), then

$$g' = \frac{1}{1/g_1 + 1/g_2} = \Phi_2(g_1, g_2) \tag{B2}$$

$$j' = 2\max(j_1, j_2) = \Psi_2(j_1, j_2)$$

Finally, with a finite probability, both branches are cut. This case can be treated separately since  $g' = 0$  and  $j'$  is undefined. In the following, we will only consider sublattices which have a non-zero conductance, and we will combine them with weights that respect the probability of each case.

Let us call  $q_n$  the probability that a lattice at the  $n$ th generation is broken, and  $p_n = 1 - q_n$  that it is conducting. At the  $(n + 1)$ th generation

$$q_{n+1} = q_n^2(2 - q_n)^2 \tag{B3}$$

Starting above the percolation threshold,  $q_0 > (3 - \sqrt{5})/2$ ,  $q_n$  converges towards 0 with  $n$ . Close to zero,  $q_n$  is approximately given by

$$q_n = 4q_{n-1}^2. \tag{B4}$$

Under iteration, we find that

$$q_n = (1/4)(4q_0)^{2^n}. \tag{B5}$$

Thus  $q_n$  approaches 0 exponentially with system size  $q_n \propto (4q_0)^L$ . For the lattices that percolates, the probability,  $P_1(n)$  and  $P_2(n)$  to be in cases 1 or 2, respectively amount to

$$P_1(n) = p_n^4 \tag{B6}$$

$$P_2(n) = 2q_n p_n^2(2 - q_n).$$

Keeping only the dominant terms gives  $P_1(n) = 1 - P_2(n)$  and  $P_2(n) = 4q_n$ .

We now have all the ingredients for studying the evolution of the conductivity and of the maximum current as a function of lattice size, or equivalently the generation. As before, we first consider the conductance of the network.

The probability to percolate tends to 1 as  $n$  increases, and thus  $P_1(n)$  also converges to 1. We also note that the function  $\Phi_1$  is contractive, and thus the distribution of conductances becomes narrower as  $n$  tends to infinity. Assuming that all the conductances of the four sublattices,  $g_i$  ( $i = 1,4$ ), are close to a common limit,  $g$ , we see that  $\Phi_1$  can be rewritten in a simpler form as

$$g' \approx \frac{1}{4}(g_1 + g_2 + g_3 + g_4). \tag{B7}$$

Thus, the central limit theorem allows us to conclude that the conductance distribution tends to a Gaussian with a width that scales as the inverse square root of the total number of bonds, or in two dimensions,  $L^{-1}$ .

Thus, we see the similarity with the previous case of a single crack in an otherwise intact lattice. The convergence of the conductance towards a well defined value, allows us to simplify the iteration relations (B1) by assuming from the beginning that all conductances are set to the same value. This approximation is the CCA that was introduced in appendix A. The function  $\Psi$  can now be written under this simplifying assumption as

$$j' = \max(j_1, j_2, j_3, j_4) = \Psi'_1(j_1, j_2, j_3, j_4)$$

$$j' = 2\max(j_1, j_2) = \Psi_2(j_1, j_2).$$
(B8)

In addition, since at the zeroth generation, all current are unity,  $j_n$  can only assume values which are integer powers of 2.

This last remarks shows that in order to characterize the statistical distribution of maximum currents in a lattice at a generation  $n$  within the CCA, it is sufficient to know the probabilities  $\pi(k, n)$  that the largest current satisfies  $j_{\max} \leq 2^k$ .

We can simply read from (B8) the recursion relations between these probabilities at two consecutive generations. The probability that the maximum of two currents satisfies  $\max(j_1, j_2) \leq 2^k$  at the  $n$ th generation is  $\pi(k, n)^2$ , whereas for four currents it is  $\pi(k, n)^4$ . Thus using the probability  $P_2(n)$  that one branch is broken, gives the recursion form

$$\pi(k, n) = P_2(n)\pi(k-1, n-1)^2 + (1 - P_2(n))\pi(k, n-1)^4$$
(B9)

with the initial conditions  $\pi(k, 0) = 1$  for all  $k \geq 0$ .

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